Generalization of n- Perfect Ring and Cotorsion dimension over on Strong n-Perfect Ring

By

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Abstract : A ring is called n-Perfect ($n \geq 0$) if every flat module has projective dimension less or equal than n. In this paper we introduce "Strong n-Perfect rings" which is in some way a generalization of the notion of "n-Perfect rings". We show that Strong n-Perfectness relates through homology with some homological dimensions of rings. We study strong n-Perfectness in some known ring construction. Finally give some examples of strong n-Perfect ring which satisfies given special conditions.

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Key words : Perfect ring , n-Perfect rings (finitstic) , strong n-Perfect ring , cotorsion dimension , Homological dimension, Pullback ring.

1. Introduction : Throughout this paper assume all rings are commutative with identity element and all modules are unitary. Let R be a ring and M be a R-module. We use $pd_R(M)$, $id_R(M)$, $fd_R(M)$ to denote the usual projective, injective and flat dimension of M. We use gldim(R) and wdim(R) , sdim(R) to denote classical global dimension, weak dimension and strong dimension of R. If R is an integral domain, we denote its quotient field by qf(R).

In [1] Bass proved that the perfect rings are those rings whose every flat module is projective. He links these rings with the finitistic projective dimension of ring.

A ring R is called strong n-Perfect then gldim(R) $\leq$ n if and only if wdim (R) $\leq$ n or in other words sdim (R) >n $\Rightarrow$ sgldim (R) >n .

The classes of rings we will define here are I some ways generalization of the notion of n-Perfect ring over strong n-Perfect ring. Let n be a positive integer. A commutative ring R is called strong n-Perfect if any R module of flat dimension less or equal than n have projective dimension less or equal than n. If n=0 then strong 0-Perfect ring are the perfect ring.

Remark : Every strong n-Perfect ring is also an n-Perfect ring.

In this paper we investigate the transfer of n-Perfect ring to strong n-Perfect property in some known ring constructions. We study the strong n-Perfect property of pullbacks and of finite direct product.

Recall that the S-finitistic projective dimension of R denoted by Sfpd (R) and defined

SFPD (R)=sup{spd_R(M)|M, R module with spd_R(M)<n}

Example : The following are equivalent for a commutative ring R :-

1. R is Perfect then R is strong Perfect.
2. R is finite direct product of local rings, each with T- nilpotent maximal ideal { for some index m, $a_1, a_2 \ldots\ldots\ldots a_m=0$}
3. FPD(R) =0 then SFPD(R)=0

   Later in [10] Jenson proved that for a ring FPD(R) $n \geq 0$ then every flat R-module has projective dimension at most n then we can conclude that if FPD(R)=n =SFPD(R) ($n \geq 0$) then every strongly flat R module has projective dimension at most n.

Enochs, Jenda and Lopez called these rings n-Perfect and thus it is strong n-Perfect. Now by homological characterized of rings by cotorsion dimension introduced by Ding and Mao.

Definition of Cotorsion dimension :

Let R be a ring the cotorsion dimension of an R module M denoted by cd_R(R) is the least positive integer n for which $Ext_R^{n+1}(F, C) = 0$, for all flat R module F.

The global cot.dim(R) denoted by c.gldim(R) is the quantity C-gldim(R)= sup.{cd_R(M)|M, R module}

If cd_R(M)=0 then it is known as 0 cotorsion modules.

Proposition 1.1 :-For a positive integer n, R is n-Perfect if and only if c-gldim(R)≤n and hence it is strong n-Perfect if C-gldim(R) $\leq n$ ⇔wdim(R) $\leq n$.

Proposition 1.2 :- For any ring R SC-gldim(R) ≤ SFPD(R).

Early in [9], Grason and Raynold defined d(R) as the supremum of the projective dimension of all flat R-modules then d(R) coincides with the s.gldim(R), they studied this invariant of rings and mentioned that Jensen had an example of a ring R that satisfy the strict inequality SC-gldim(R) ≤SFPD(R).

Now we give some general results on the global inequality gldim(R)≤c-gldim(R)+wdim(R) establish by Ding and Mao in [5] to the finitistic dimension and thus we represent this inequality for strong ring

S.gldim(R) ≤ Sc-gldim(R)+S.dim(R)

S.gldim(R)<n+n⇒ S.gldim(R) <2n

since for strong n-perfect ring S.c-gldim(R)≤ n

S.gldim(R)=s.dim(R) ≤n⇔gldim(R)≤n⇔wdim)≤ n

In sec. 2 we give general result on the global cotorsion dimension of rings is used to relate the strong global dimension over weak and the global dimension.

In sec. 3 we investigate strong n-Perfectness in some known ring construction such that we compute the strong global cotorsion dimension of polynomial rings, reducible to polynomial rings, finite direct product of rings and D+M rings. This study allows us to give various example of Strong n-Perfect rings satisfying special conditions given in sec.4.

2. General Results :

Theorem 2.1 : For any ring R the following inequalities S.gldim(R) ≤ S.cgldim(R)+ sdim(R) holds true.

Proof : Since gldim(R) ≤ c.gldim(R)+ wdim(R)

c.gldim(R) ≤ n⇔wdim(R) ≤ n

S.gldim(R)= Sdim(R) $\leq n$ ⇔gldim(R) ≤ wdim(R) $\leq n$

In particular

1. If S.cgldim(R)=0 i.e R is Strong Perfect then Sdim(R)=Sgldim(R) $\leq n$ since R is perfect then c.gldim(R)=0⇒wdim(R)= gldim(R) $\leq n$
2. If Sdim(R)=0 {wdim(R)=0} i.e R is von Neumann regular then Scgldim(R)= Sgldim(R).

Now we generalize this result to the finitistic projective and flat dimensions. Recall that the strongly finitistic flat dimension of R SFFD(R) is defined as follows SFFD(R) =sup{sfd_R(M)|M, R module with sfd_R(M)<∞ }

Theorem 2.2 : For any ring R the following inequalities SFFD(R) ≤ SFPD(R) ≤ Scgldim(R)+ SFFD(R) hold true.

Proof : To prove the inequality SFFD(R) ≤ SFPD(R) we can assume that SFPD(R)=n is finite. Consider R module M with finite flat dimension from Jensen's Proposition 1.2 M has also finite projective dimension which is at most n then we have sfd_R(M) ≤spd_R(M) ≤ n. It means that SFFD(R) ≤ SFPD(R).

Now we prove the second inequality for that we can assume that S.cgldim(R)=n and SFFD(R)=m are finite. Consider an R module M with finite projective dimension then it has finite flat dimension which is at most m then there exist an exact sequence of R modules

$$0 \to F \to P_{m-1} \to \ldots\ldots\ldots\ldots \to P_0 \to N \to 0$$

Where P_i are projective and F is flat. From proposition 1.1 spdR(F) ≤ n. finally using the above sequence we get spdR(M) ≤n+m. This completes the proof.

Proposition 2.3 : If a ring R satisfies SFFD(R)=1 and S.cgldim(R)<SFPD(R)< ∞ then SFPD(R) =S.cgldim(R)+1 .

Proof : Let SFPD(R)=n<∞ for an integer n≥1 then there is an R module M that satisfies

spdR(M)=n. hence for a short exact sequence of R modules 0→F→P→M→0 where P is projective and F is flat. Since SFFD(R)=1 then spd_R(F)=n-1 then equality holds since

S.cgldim(R)<SFPD(R).

Corollary : If a ring R satisfies SFFD(R)=m and S.cgldim(R)<SFPD(R)< ∞ then SFPD(R) =S.cgldim(R)+m .

Recall that the S finitistic injective dimension of a ring R is denoted by SFID(R) and defined by

SFID(R)= sup{sidR(M)|M, R module with sid_R(M)< ∞}.

Similarly we can define the finitistic cotorsion dimension of a ring R denoted by SFCD(R) and defined by SFCD(R)= sup{ scd_R(M)|m, R module with scd_R(M) <∞}

Theorem 2.4 : For any ring R with finite strong dimension the following inequalities

SFCD(R) ≤ SFID(R) ≤ SFCD(R) +S.dim(R)

It involves lemma which relates the cotorsion dimension and the injective dimension of modules.

Lemma 2.5 : Let R be a ring for any R module M the following inequalities

Scd(M)≤ Sid(M) ≤Scd(M)+Sdim(R) hold and true.

Proof : First we prove the lemma we can assume that Scd(M)=m and Sdim(R)=n are finite. Let N be any R module and consider an exact sequence $0 \to F \to P_{n-1} \to \ldots\ldots\ldots \to P_0 \to N \to 0$ where Pi are projective modules and then F is a flat module. Since Sdim(R)=n.

We have Ext_R^{m+n+1} (N,M)≅ Ext_R^{m+1}(F,M)=0. Since Scd_R(M)=n therefore Sid_R(M) ≤ m+n

Now we prove Theorem 2.4

Let SFID(R)=n is finite. M be an R module with finite cotorsion dimension from lemma

Sid(M) $\leq$ Scd(M)+Sdim(R) which is finite then Sid(M) is finite and so Scd(M) $\leq$Sid(M) $\leq n$

We prove SFID(R) $\leq$SFCD(R)+ Sdim(R)

Let Sdim(R)=n, SFCD(R)=m are finite. Let M be an R module with finite injective module dimension then from lemma Sid(M) $\leq$Scd(M)+Sdim(R) $\leq$m+n.

3. Strong n-Perfectness in Ring Construction

Firstly we compute the strong global cotorsion dimension of polynomial ring and family of polynomial rings.

Theorem 3.1 ; Let R[X1,X2………Xn] be a polynomial ring in n indeterminates over a ring R then for a positive integer m , R is m strong perfect if and only if R[X_1,X_2………..X_n] is m+n Perfect strongly.

Scgldim(R[X_1,X_2,……..X_n]) = Scgldim(R)+n

Proposition 3.2 : Let (R_i) where i=1,2,3,…….m be a family of ring S then $\prod_{i=1}^{m}R_i$ is a strong n Perfect ring if and only if R_i is a strong n Perfect ring for each i=1,2,3……..m.

Proof : Let (R_i) where i=1,2,3……m be a family of rings and Mi be an R_i Module for i=1,2,3…..m. we have

SpdR_1xR_2(M_1xM_2) = Sup{ SpdR_1(M_1);SpdR_2(M_2)}………………………………………….(1)

SfdR_1xR_2(M_1xM_2) =Sup{ SfdR_1(M_1); SfdR_2(M_2)}…………………………………………….(2)

Sid_{R1xR2}(M_1xM_2) = Sup{SidR_1(M_1); SidR_2(M_2)}…………………………………………….(3)

We prove theorem by induction method. Let it is true for m=2. Let R_1 and R_2 be two rings such that R_1xR_2 is a strong n-Perfect and let M_1 be an R_1 module such that SfdR_1(M_1) $\leq n$, let M_2 be an R_2 module such that SfdR_2(M_2) $\leq n$ then Sfd_{R1xR2}(M_1xM_2)= Sup{sfdR_1(M_1); SfdR_2(M_2)} by (2) .

Spd_{R1xR2}(M_1xM_2) $\leq n$ since R_1xR_2 is strong n-Perfect thus Spd_{R1xR2}(M_1xM_2)=Sup{SpdR_1(M_1);SpdR_2(M_2)} therefore SpdR_1(M_1) $\leq n$ and SpdR_2(M_2) $\leq n$ and so R_1 and R_2 are strong n perfect rings thus Sid_{R1xR2}(M_1xM_2) $\leq n$ since R_1xR_2 is strong n-Perfect therefore SidR_1xR_2(M_1xM_2)=Sup{SidR_1(M_1); SidR_2(M_2)} by (3) hence SidR_1(M_1)$\leq n$ and SidR_2(M_2) $\leq n$.

Conversely : let R_1 and R_2 be two strong n-Perfect rings. Let M_1xM_2 be an R_1xR_2 module where Mi is an Ri module for each i=1,2 such that SfdR1xR2(M_1xM_2) $\leq n$ thus SfdR_1(M_1) $\leq n$, SfdR_2(M_2) $\leq n$ and SpdR_1xR_2(M_1xM_2) $\leq n$ thus SpdR_1(M_1) $\leq n$

SpdR_2(M_2) $\leq n$ and SidR1xR2(M1xM2) $\leq n$ thus SidR_1(M_1) $\leq n$, SidR_2(M_2) $\leq n$ because R_1 and R_2 are strong n-Perfect ring and so R_1x R_2 is a strong n-Perfect ring.

Cotorsion dimension under change of rings

Theorem 4.1 : let β: R_i⟶S_i be a surjective ring homomorphism where R_i and S_i be family of rings R and S , i=1,2,3….m then

1. If M_{Si} is a rinht Strong S_i module then Scd(M_{Ri}) $\leq$Scd(M_{Si})moreover if S_{Ri} is a flat right R module then Scd(M_{Si})= Scd(M_{Ri})
2. If SRi is a flat right Ri module and MRi is a cotorsion right Ri module then Hom_{Ri}(Si,Mi) is a cotorsion right right Si modle and hence a cotorsion right Ri module where i=1,2,3,……..m

Proof :

1. We may assume Scd(M_{Si})=n<∞ there exist an exact sequence

0⟶Mi⟶ $C_i^0 \longrightarrow C_i^1 \longrightarrow C_i^2 \longrightarrow$…………..⟶ $C_i^{n-1} \longrightarrow C_i^n$ ⟶0

Where C^j is a cotorsion right Si module , j=1,2,……n each C^j is also cotorsion as a right Ri module so Scd(M_{Ri}) $\leq n$.

If S_{Ri} is a flat right Si module we claim Scd(M_{Si}) ≤Scd(M_{Ri}). We assume Scd(MRi)=n<∞. Let Fi be a flat right Si module then Fi is a fat right Ri module thus Ext_{Si}^{n+1}(F_{Si}, M_{Si})= Ext_{Ri}^{n+1}(F_{Ri},M_{Ri})=0 therefore Scd(M_{Si}) $\leq n$ and hence Scd(MSi)=Scd(M_{Ri})

2 by hypothesis Ext_{Ri}^{1}(Si,Mi) =0 let X be a flat right Si module then X is a flat right Ri moue thus Ext_{Si}^{n+1}(X,HomR(Si,Mi)= Ext_{Ri}^{n+1}(X,Mi)=0

Therefore Hom_R(Si,Mi) is a cotorion right Si module and hence a cotorsion right Ri module.

Corollary 4.2 : Let π: Ri⟶Si bea surjective ring homomorphism and SRi is a flat right Ri module then r.cotD(Si) ≤ r.cotD(Ri)

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